GELFAND-KIRILLOV DIMENSIONS OF THE $\mathbb{Z}^2$-GRADED OSCILLATOR REPRESENTATIONS OF $\mathfrak{sl}(n)$

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ABSTRACT. We find an exact formula of Gelfand-Kirillov dimensions for the infinite-dimensional explicit irreducible $\mathfrak{sl}(n,\mathbb{F})$-modules that appeared in the $\mathbb{Z}^2$-graded oscillator generalizations of the classical theorem on harmonic polynomials established by Luo and Xu. Three infinite subfamilies of these modules have the minimal Gelfand-Kirillov dimension. They contain weight modules with unbounded weight multiplicities and completely pointed modules.

Key Words: Gelfand-Kirillov dimension; Highest-weight module; Oscillator representation; Universal enveloping algebra.

1. INTRODUCTION

In 1960s, Gelfand-Kirillov [8] introduced a quantity to measure the rate of growth of an algebra in terms of any generating set, which is now known as Gelfand-Kirillov dimension. In 1970s, Vogan [23] and Joseph [13] used the Gelfand-Kirillov dimension to measure the size of the infinite-dimensional modules. Gelfand-Kirillov dimension has been an important invariant in the theory of algebras over a field for the past forty years. Although it is rarely exact, it has the advantage over Krull dimension of being both symmetric and ideal invariant. It has been applied successfully to enveloping algebras, Weyl algebras, and more generally to filtered and graded algebras. But in general, the Gelfand-Kirillov dimension of an infinite-dimensional module is not easy to compute.

A module of a finite-dimensional simple Lie algebra is called a *weight module* if it is a direct sum of its weight subspaces. Let M be an irreducible highest-weight module for a finite-dimensional simple Lie algebra $\mathfrak{g}$. Then M is naturally a weight module with finite-dimensional weight subspaces. Denote by d_M its Gelfand-Kirillov dimension. We fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Delta \subset \mathfrak{h}^*$ and a set of positive roots $\Delta_+ \subset \Delta$. Let ρ be half the sum of all positive roots. Suppose that β is the highest root. It is well known that $d_M = 0$ if and only if M is finite-dimensional, in which case irreducible modules are classified by the highest-weight theory. From Vogan [24] and Wang [26], we know that the next smallest integer occurring is $d_M = (\rho, \beta^\vee)$. We call them the *minimal Gelfand-Kirillov dimension module*. These small modules are of great interest in representation theory. A general introduction can be found in Vogan [24]. Fernando [7] showed that the only simple Lie algebras which have simple torsion-free modules (or cuspidal modules, i.e.,weight modules on which all root vectors of the Lie algebra act bijectively) are those of types A_n or C_n. He also showed that these finitely generated torsion-free $\mathcal{U}(\mathfrak{g})$-modules have the minimal Gelfand-Kirillov dimension. A similar result was independently obtained by Futorny [6]. In the last 20 years cuspidal modules have been extensively studied by Benkart, Britten, Grantcharov, Hooper, Khomenko, Lemire, Mathieu, Mazorchuk, Serganova and others (see e.g. [3],[4], [9],[10], [18]).

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Benkart, Britten and Lemire [1] generalized Fernando's work. They proved that infinite-dimensional weight modules, whose weight-subspace dimensions are bounded, have the minimal Gelfand-Kirillov dimension and exist only for finite-dimensional simple Lie algebras of types A_n and C_n. Britten and Lemire [5] described all weights $\omega \in \mathfrak{h}^*$ such that the corresponding irreducible highest-weight $\mathfrak{sp}(2n,\mathbb{C})$-module $L(\omega)$ has bounded multiplicities. Sun [22] classified genuine irreducible lowest-weight modules of $\mathfrak{sp}(2n,\mathbb{C})$ with minimal Gelfand-Kirillov dimension. But other examples of minimal Gelfand-Kirillov dimension modules are not well-known.

In classical harmonic analysis, a fundamental theorem says that the spaces of homogeneous harmonic polynomials are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Bases of these irreducible modules can be obtained easily (e.g., cf. [27]). The algebraic beauty of the above theorem is that Laplace equation characterizes the irreducible submodules of the polynomial algebra and the corresponding quadratic invariant gives a decomposition of the polynomial algebra into a direct sum of irreducible submodules, namely, the complete reducibility. Algebraically, this can be interpreted as an $(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{o}(n,\mathbb{R}))$ Howe duality. Recently Luo and Xu [17] established the $\mathbb{Z}^2$-graded oscillator generalizations of the above theorem for $\mathfrak{sl}(n,\mathbb{F})$, where the irreducible submodules are $\mathbb{Z}^2$-graded homogeneous polynomial solutions of deformed Laplace equations. An important feature of the irreducible modules in [17] is that the corresponding representation formulas are simple and bases are easily given. In fact, these are explicit infinite-dimensional highest-weight irreducible $\mathfrak{sl}(n,\mathbb{F})$-modules. The goal of this paper is to find an exact formula of Gelfand-Kirillov dimension for them. It turns out that these Gelfand-Kirillov dimensions are independent of the double grading and four infinite subfamilies of these modules have the minimal Gelfand-Kirillov dimension. In general, our result will be useful in studying the $\mathfrak{sl}(n,\mathbb{F})$-modules with a given Gelfand-Kirillov dimension. Below we give more detailed technical introduction.

Throughout this paper, the base field $\mathbb{F}$ has characteristic 0. For convenience, we will use the notion $\overline{i,i+j}=\{i,i+1,i+2,...,i+j\}$ for integers i and j with $i\leq j$. Denote by $\mathbb{N}$ the additive semigroup of nonnegative integers. Let $E_{r,s}$ be the square matrix with 1 as its (r,s)-entry and 0 as the others. Moreover, we always assume that $n\geq 2$ is an integer. Denote $\mathscr{B}=\mathbb{F}[x_1,...,x_n,y_1,...,y_n]$. Fix $n_1,n_2\in\overline{1,n}$ with $n_1\leq n_2$. Changing operators $\partial_{x_r}\mapsto -x_r,\ x_r\mapsto\partial_{x_r}$ for $r\in\overline{1,n_1}$ and $\partial_{y_s}\mapsto -y_s,\ y_s\mapsto\partial_{y_s}$ for $s\in\overline{n_2+1,n}$ in the canonical oscillator representation $E_{i,j}|_{\mathscr{B}}=x_i\partial_{x_j}-y_j\partial_{y_i}$ for $i,j\in\overline{1,n}$, we get the following non-canonical oscillator representation of $\mathfrak{sl}(n,\mathbb{F})$ on $\mathscr{B}$ determined by

$$E_{i,j}|_{\mathscr{B}}=E^x_{i,j}-E^y_{j,i}\qquad\text{for } i,j\in\overline{1,n}\tag{1.1}$$

with

$$E^x_{i,j}|_{\mathscr{B}}=\begin{cases}-x_j\partial_{x_i}-\delta_{i,j} & \text{if } i,j\in\overline{1,n_1};\\ \partial_{x_i}\partial_{x_j} & \text{if } i\in\overline{1,n_1},\ j\in\overline{n_1+1,n};\\ -x_ix_j & \text{if } i\in\overline{n_1+1,n},\ j\in\overline{1,n_1};\\ x_i\partial_{x_j} & \text{if } i,j\in\overline{n_1+1,n}\end{cases}\tag{1.2}$$

and

$$E^y_{i,j}|_{\mathscr{B}}=\begin{cases}y_i\partial_{y_j} & \text{if } i,j\in\overline{1,n_2};\\ -y_iy_j & \text{if } i\in\overline{1,n_2},\ j\in\overline{n_2+1,n};\\ \partial_{y_i}\partial_{y_j} & \text{if } i\in\overline{n_2+1,n},\ j\in\overline{1,n_2};\\ -y_j\partial_{y_i}-\delta_{i,j} & \text{if } i,j\in\overline{n_2+1,n}.\end{cases}\tag{1.3}$$

The related variated Laplace operator becomes

$$\mathscr{D} = -\sum_{i=1}^{n_1} x_i \partial_{y_i} + \sum_{r=n_1+1}^{n_2} \partial_{x_r} \partial_{y_r} - \sum_{s=n_2+1}^{n} y_s \partial_{x_s} \tag{1.4}$$

and its dual

$$\eta = \sum_{i=1}^{n_1} y_i \partial_{x_i} + \sum_{r=n_1+1}^{n_2} x_r y_r + \sum_{s=n_2+1}^{n} x_s \partial_{y_s}. \tag{1.5}$$

Set

(1.6)

$$\mathscr{B}_{\langle \ell_1, \ell_2 \rangle} = \text{Span}\{x^\alpha y^\beta \mid \alpha, \beta \in \mathbb{N}^n, \sum_{r=n_1+1}^{n} \alpha_r - \sum_{i=1}^{n_1} \alpha_i = \ell_1, \sum_{i=1}^{n_2} \beta_i - \sum_{r=n_2+1}^{n} \beta_r = \ell_2\}$$

for $\ell_1, \ell_2 \in \mathbb{Z}$. Define

$$\mathscr{H}_{\langle \ell_1, \ell_2 \rangle} = \{f \in \mathscr{B}_{\langle \ell_1, \ell_2 \rangle} \mid \mathscr{D}(f) = 0\}. \tag{1.7}$$

Luo and Xu [17] proved that for any $\ell_1, \ell_2 \in \mathbb{Z}$ such that $\ell_1 + \ell_2 \leq n_1 - n_2 + 1 - \delta_{n_1,n_2}$, $\mathscr{H}_{\langle \ell_1, \ell_2 \rangle}$ is an irreducible highest-weight $\mathfrak{sl}(n, \mathbb{F})$-module. Moreover, the homogeneous subspace $\mathscr{B}_{\langle \ell_1, \ell_2 \rangle} = \bigoplus_{m=0}^{\infty} \eta^m (\mathscr{H}_{\langle \ell_1 - m, \ell_2 - m \rangle})$ is a direct sum of irreducible submodules. In some special cases, they obtained more general results. The following is the main theorem of this paper.

Theorem 1.1. *For any $\ell_1, \ell_2 \in \mathbb{Z}$, if $\mathfrak{sl}(n, \mathbb{F})$-module $\mathscr{H}_{\langle \ell_1, \ell_2 \rangle}$ is irreducible, then it has the Gelfand-Kirillov dimension*

(1.8)

$$d = \begin{cases} 2n-2, & \text{if } 1 < n_1 < n_2 < n-1 \text{ or } 3 \leq n_1 = n_2 \leq n-3, n \geq 7; \\ 2n-3, & \text{if } 1 = n_1 < n_2 < n \text{ or } n_1 < n_2 = n-1 \text{ or } n_1 = n_2 = 3 < n = 6; \\ 2n-4, & \text{if } n_1 = n_2 = 2 < n-1 \text{ or } 1 < n_1 = n_2 = n-2; \\ n, & \text{if } 1 < n_1 = n_2 < n-1; \\ n-1, & \text{otherwise.} \end{cases}$$

We find that some of these irreducible highest-weight $\mathfrak{sl}(n, \mathbb{F})$-modules have the minimal Gelfand-Kirillov dimension. The result is as follows.

Corollary 1.1. *An irreducible $\mathfrak{sl}(n, \mathbb{F})$-module $\mathscr{H}_{\langle \ell_1, \ell_2 \rangle}$ has the minimal Gelfand-Kirillov dimension $n-1$ if and only if*

(1) $n_1 < n_2 = n$.
(2) $n_1 = n_2 = 1$.
(3) $n_1 = n_2 = n-1$.

From Benkart-Britten-Lemire [1], an infinite-dimensional weight module is called *completely pointed* if its weight spaces are all one-dimensional. Using the results on modules for Weyl algebras, they provided the whole list of simple infinite-dimensional completely pointed modules. These modules are described in terms of multiplication and differentiation operators on "polynomials". Comparing with their result [1], we can give different realizations for some of these simple infinite-dimensional completely pointed modules.

Corollary 1.2. *An irreducible $\mathfrak{sl}(n, \mathbb{F})$-module $\mathscr{H}_{\langle \ell_1, \ell_2 \rangle}$ (with highest weight vector v_λ of weight λ) is completely pointed if and only if*

(1) $n_1+1<n_2=n$,
$\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle m_1,0\rangle}$, *with* $v_\lambda=x_{n_1+1}^{m_1}$, $\lambda=-(m_1+1)\lambda_{n_1}+m_1\lambda_{n_1+1}$, $m_1\in\mathbb{N}$
or $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle -m_1,0\rangle}$, *with* $v_\lambda=x_{n_1}^{m_1}$, $\lambda=m_1\lambda_{n_1-1}-(m_1+1)\lambda_{n_1}$, $m_1\leq n-n_1-2$ *or* $m_1\geq n-n_1-1$.

(2) $n_1+1=n_2=n$,
$\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle -m_1,m_2\rangle}$, *with* $v_\lambda=x_1^{m_1}y_2^{m_2}$, $\lambda=(m_2-m_1-1)\lambda_1$, $m_i\in\mathbb{N}$ *and* $m_2\leq m_1$, $n=2$
or $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle -m_1,0\rangle}$, *with* $v_\lambda=x_{n-1}^{m_1}$, $\lambda=m_1\lambda_{n-2}-(m_1+1)\lambda_{n-1}$, $m_1\in\mathbb{N}$
or $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle m,0\rangle}$, *with* $v_\lambda=x_{n-1}^{m}$, $\lambda=m\lambda_{n-2}-(m+1)\lambda_{n-1}$, $m\in\mathbb{Z}$.

(3) $n_1=n_2=1$,
$\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle -m_1,0\rangle}$, *with* $v_\lambda=x_1^{m_1}$, $\lambda=-(m_1+2)\lambda_1$, $m_1\in\mathbb{N}$
or $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle m_1+1,-m_1-1\rangle}$, *with* $v_\lambda=\zeta_2^{m_1+1}$, $\lambda=-(m_1+3)\lambda_1$, $m_1\in\mathbb{N}$, $n=3$.

(4) $n_1=n_2=n-1$,
$\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle 0,-m_2\rangle}$, *with* $v_\lambda=y_n^{m_2}$, $\lambda=-(m_2+2)\lambda_{n-1}$, $m_2\in\mathbb{N}$
or $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}=\mathscr{H}_{\langle -m_2-1,m_2+1\rangle}$, *with* $v_\lambda=\zeta_1^{m_2+1}$, $\lambda=-(m_2+3)\lambda_2$, $m_2\in\mathbb{N}$, $n=3$.

Benkart-Britten-Lemire [1] showed that when infinite-dimensional simple $\mathfrak{sl}(n,\mathbb{F})$-modules have bounded weight multiplicities, they must have the minimal GK-dimension $n-1$. From our formula and Luo-Xu [17], we can easily find that the converse doesn't hold. For example, when $n_1=n_2=1<n$, the irreducible highest-weight module $\mathscr{H}_{\langle -1,-1\rangle}$ has a highest weight vector x_1y_2 of weight $-4\lambda_1+\lambda_2$. But it doesn't have bounded weight multiplicities.

In Section 2 we recall the basic definition of the Gelfand-Kirillov dimension. In Section 3, we give the formula on the Gelfand-Kirillov dimensions of these highest-weight modules $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}$. In Section 4, we give a proof for our formula. The proof is given in a case-by-case way. In Section 5, we give a corollary about the $\mathfrak{sl}(n,\mathbb{F})$-modules which have the minimal Gelfand-Kirillov dimension.

2. Preliminaries on Gelfand-Kirillov Dimension

In this section we recall the definition and some properties of the Gelfand-Kirillov dimension. The details can be found in Refs.[2, 12, 15, 20, 23, 25].

Definition 2.1. *Let A be an algebra (not necessarily associative) generated by a finite-dimensional subspace V. Let V^n denote the linear span of all products of length at most n in elements of V. The* Gelfand-Kirillov dimension *of A is defined by:*

$$GKdim(A)=\limsup_{n\to\infty}\frac{\log\dim(V^n)}{\log n}$$

Remark 2.1. *It is well-known that the above definition is independent of the choice of the finite dimensional generating subspace V (see Ref.[2, 15]). Clearly $GKdim(A)=0$ if and only if $\dim(A)<\infty$.*

The notion of Gelfand-Kirillov dimension can be extended for left A-modules. In fact, we have the following definition.

Definition 2.2. *Let A be an algebra (not necessarily associative) generated by a finite-dimensional subspace V. Let M be a left A-module generated by a finite-dimensional*

subspace M_0. Let V^n denote the linear span of all products of length at most n in elements of V. The Gelfand-Kirillov dimension $GKdim(M)$ *of M is defined by*

$$GKdim(M) = \limsup_{n\to\infty} \frac{\log \dim(V^n M_0)}{\log n}.$$

In particular, let $\mathfrak{g}$ be a complex Lie algebra. Let $A = \mathcal{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, with the standard filtration given by $A_n = \mathcal{U}_n(\mathfrak{g})$, the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by products of at most n-elements of $\mathfrak{g}$. By the Poincaré-Birkhoff-Witt theorem (see Knapp [14, Prop. 3.16]), the graded algebra gr$(\mathcal{U}(\mathfrak{g}))$ is canonically isomorphic to the symmetric algebra $S(\mathfrak{g})$. Suppose M is a $\mathcal{U}(\mathfrak{g})$-module generated by a finite-dimensional subspace M_0. We set $M_n = \mathcal{U}_n(\mathfrak{g})M_0$. Denote gr$M = \bigoplus_{n=0}^{\infty}$ gr$_n M$, where gr$_n M = M_n/M_{n-1}$. Then gr$M$ becomes a graded $S(\mathfrak{g})$-module. We denote $\dim(M_n)$ by $\varphi_M(n)$. Then we have the following lemma.

Lemma 2.1. (Hilbert-Serre [29, Chapter VII. Th.41] and [28])

(1) *With the notations as above, there exists a unique polynomial $\tilde{\varphi}_M(n)$ such that $\varphi_M(n) = \tilde{\varphi}_M(n)$ for large n. The leading term of $\tilde{\varphi}_M(n)$ is*

$$\frac{c(M)}{(d_M)!} n^{d_M},$$

where $c(M)$ is an integer.

(2) *The degree d_M of this polynomial $\tilde{\varphi}_M(n)$ is equal to the dimension of the* associated variety

$$\mathscr{V}(M) = \{X \in \mathfrak{g}^* | p(X) = 0, \forall p \in \mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}M)\},$$

where $\mathrm{Ann}_{S(\mathfrak{g})}(\mathrm{gr}M) = \{D \in S(\mathfrak{g}) | Dv = 0, \forall v \in \mathrm{gr}M\}$ is the annihilator ideal of $\mathrm{gr}M$ in $S(\mathfrak{g})$, and $S(\mathfrak{g})$ is identified with the polynomial ring over $\mathfrak{g}^*$ through the Killing form of $\mathfrak{g}$.

Remark 2.2. *From the definition of Gelfand-Kirillov dimension, we know*

$$GKdim(M) = \limsup_{n\to\infty} \frac{\log \dim(U_n(\mathfrak{g})M_0)}{\log n} = \limsup_{n\to\infty} \frac{\log \tilde{\varphi}_M(n)}{\log n} = d_M = \dim \mathscr{V}(M).$$

Example 2.1. *Let $M = \mathbb{C}[x_1, ..., x_k]$. Then M is an algebra generated by the finite-dimensional subspace $V = Span_{\mathbb{C}}\{x_1, ..., x_k\}$. So $M_n = V^n = \bigoplus_{0\le q\le n} P_q[x_1, ..., x_k]$ is the subset of homogeneous polynomials of degree $\le n$. Then*

$$\begin{aligned}\varphi_M(n) &= \sum_{0\le q\le n} \dim_{\mathbb{C}}(P_q[x_1, ..., x_k]) \\ &= \sum_{0\le q\le n} \binom{k+q-1}{q} \\ &= \binom{k+n}{n} \\ &= \frac{n^k}{k!} + O(n^{k-1}).\end{aligned}$$

Then we have $GKdim(M) = k$.

3. A FORMULA ON THE GELFAND-KIRILLOV DIMENSION

We keep the same notations with the introduction. Write

$$\zeta_1 = x_{n_1-1}y_{n_1} - x_{n_1}y_{n_1-1}, \ \ \zeta_2 = x_{n_1+1}y_{n_1+2} - x_{n_1+2}y_{n_1+1}. \tag{3.1}$$

From Luo-Xu [17], the followings are detailed irreducible highest weight $\mathfrak{sl}(n,\mathbb{F})$-modules:

(1) $n_1+1<n_2<n$. For $m_1, m_2 \in \mathbb{N}$ with $m_1+m_2 \geq n_2-n_1-1$, $\mathscr{H}_{\langle -m_1,-m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n_2+1}^{m_2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_2-m_1 \geq n_2-n_1-1$, $\mathscr{H}_{\langle m_1,-m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1+1}^{m_1}y_{n_2+1}^{m_2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_1-m_2 \geq n_2-n_1-1$, $\mathscr{H}_{\langle -m_1,m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n_2}^{m_2}$.

(2) $n_1+1<n_2=n$. For $m_1, m_2 \in \mathbb{N}$, $\mathscr{H}_{\langle m_1,m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1+1}^{m_1}y_{n}^{m_2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_1 \leq n-n_1-2$ or $m_2-m_1 \leq n_1-n+1$, $\mathscr{H}_{\langle -m_1,m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n}^{m_2}$.

(3) $n_1+1=n_2<n$. For $m_1, m_2 \in \mathbb{N}$, $\mathscr{H}_{\langle -m_1,-m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n_1+2}^{m_2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_2-m_1 \geq 0$, $\mathscr{H}_{\langle m_1,-m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1+1}^{m_1}y_{n_1+2}^{m_2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_1-m_2 \geq 0$, $\mathscr{H}_{\langle -m_1,m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n_1+1}^{m_2}$.

(4) $n_1+1=n_2=n$. For $m_1, m_2 \in \mathbb{N}$ with $m_2 \leq m_1$, $\mathscr{H}_{\langle -m_1,m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n-1}^{m_1}y_{n}^{m_2}$. Moreover, $\mathscr{H}_{\langle m,0\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector x_{n-1}^{m}. For $m_1, m_2 \in \mathbb{N}+1$, $\mathscr{H}_{\langle m_1,m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector
$\eta^{m_1+m_2}(x_{n-1}^{m_2}y_n^{-m_1})$.

(5) $n_1=n_2<n-1$. Let $m_1, m_2 \in \mathbb{N}$. The subspace $\mathscr{H}_{\langle -m_1,-m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n_1+1}^{m_2}$. The subspace $\mathscr{H}_{\langle m_1+1,-m_2-m_1-1\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $y_{n_1+1}^{m_2}\zeta_2^{m_1+1}$. If $n_1 \geq 2$, the subspace $\mathscr{H}_{\langle -m_1-m_2-1,m_2+1\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}\zeta_1^{m_2+1}$.

(6) $n_1=n_2=n-1$. Let $m_1, m_2 \in \mathbb{N}$. The subspace $\mathscr{H}_{\langle -m_1,-m_2\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}y_{n_1+1}^{m_2}$. If $n \geq 3$, the subspace
$\mathscr{H}_{\langle -m_1-m_2-1,m_2+1\rangle}$ is an infinite-dimensional irreducible module with a highest-weight vector $x_{n_1}^{m_1}\zeta_1^{m_2+1}$.

(7) $n_1=n_2=n$. Let $m_1, m_2 \in \mathbb{N}$. The subspace $\mathscr{H}_{\langle -m_1-m_2,m_2\rangle}$ is finite-dimensional and is an irreducible module with a highest-weight vector $x_{n}^{m_1}\zeta_1^{m_2}$.

Then we have our main theorem, i.e. Theorem 1.1.

4. PROOF OF THE MAIN THEOREM

Now we want to compute the Gelfand-Kirillov dimension of the $\mathfrak{sl}(n,\mathbb{F})$-module $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}$ for all cases. Suppose its highest weight vector is v_λ with highest weight λ.

We simply write $E_{i,j}|_{\mathscr{B}}$ as $E_{i,j}$. Take

$$\mathfrak{h} = \sum_{i=1}^{n-1} \mathbb{F}(E_{i,i} - E_{i+1,i+1}) \tag{4.1}$$

as a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{F})$ and the subspace spanned by positive root vectors:

$$\mathfrak{sl}(n, \mathbb{F})_+ = \sum_{1 \leq i < j \leq n} \mathbb{F}E_{i,j}. \tag{4.2}$$

Correspondingly, we have

$$\mathfrak{sl}(n, \mathbb{F})_- = \sum_{1 \leq i < j \leq n} \mathbb{F}E_{j,i}. \tag{4.3}$$

From the highest-weight module theorem we know that $\mathscr{H}_{\langle \ell_1, \ell_2 \rangle} = \mathcal{U}(\mathfrak{g})v_\lambda = \mathcal{U}(\mathfrak{g}_-)v_\lambda$. In the following we will compute the Gelfand-Kirillov dimension of $\mathcal{U}(\mathfrak{g}_-)v_\lambda$ in a case-by-case way.

Firstly we need the following two well-known lemmas.

Lemma 4.1. (Multinomial theorem)
Let n, m be two positive integers, then

$$\left| \{(k_1, k_2, ..., k_m) \in \mathbb{N}^m | \sum_{i=1}^{m} k_i = n\} \right| = \binom{n+m-1}{m-1}. \tag{4.4}$$

Lemma 4.2. *Let p, n be two positive integers, then*

$$\sum_{i=0}^{n} i^p = \frac{(n+1)^{p+1}}{p+1} + \sum_{k=1}^{p} \frac{B_k}{p-k+1} \binom{p}{k} (n+1)^{p-k+1} \approx \frac{n^{p+1}}{p+1}, \tag{4.5}$$

where B_k denotes a Bernoulli number.

From these two lemmas, we can get the following several propositions.

Proposition 4.1. *Let $k \in \mathbb{N}$ and we denote* $M_k = \left\{ \prod_{\substack{1 \leq i \leq n_1 \\ n_1+1 \leq t \leq n}} (x_i x_t)^{p_{it}} | \sum_{\substack{1 \leq i \leq n_1 \\ n_1+1 \leq t \leq n}} p_{it} = k, p_{it} \in \mathbb{N} \right\}$.
Then

$$a_k = \dim Span_{\mathbb{R}} M_k = \binom{n_1 + k - 1}{k} \binom{n - n_1 + k - 1}{k} \approx a k^{n-2}, \tag{4.6}$$

for some constant a.

Proof. From the definition of M_k, we know that all the elements in M_k are monomials and they must form a basis for $Span_{\mathbb{R}} M_k$. Thus

$$\begin{aligned} a_k =& \dim Span_{\mathbb{R}} M_k = \# \left\{ \prod_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} (x_i x_t)^{p_{it}} | \sum_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} p_{it} = k, p_{it} \in \mathbb{N} \right\} \\ =& \# \left\{ \prod_{1\leq i\leq n_1} (x_i)^{\sum_{n_1+1\leq t\leq n} p_{it}} \prod_{n_1+1\leq t\leq n} (x_t)^{\sum_{1\leq i\leq n_1} p_{it}} | \sum_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} p_{it} = k, p_{it} \in \mathbb{N} \right\} \\ =& \binom{n_1+k-1}{k}\binom{n-n_1+k-1}{k} \approx a k^{n-2}, \text{ for some constant } a. \end{aligned}$$

□

Proposition 4.2. *Let $k \in \mathbb{N}$ and we denote* $N_k = \left\{ \prod_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} (x_i x_t - y_i y_t)^{h_{it}} | \sum_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} h_{it} = k, h_{it} \in \mathbb{N} \right\}$. *Then we have*

$$d_k = \dim Span_{\mathbb{R}} N_k \approx \begin{cases} c_0 k^{n-2}, & \text{if } n_1 = 1 \text{ or } n_1 = n-1; \\ c_1 k^{2n-5}, & \text{if } n_1 = 2 < n-1 \text{ or } 1 < n_1 = n-2; \\ c_2 k^{2n-4}, & \text{if } n_1 = 3 < n = 6; \\ c_3 k^{2n-3}, & \text{if } 3 \leq n_1 \leq n-3, n \geq 7. \end{cases}$$

Here c_0, c_1, c_2 and c_3 are some positive constants which are independent of k.

Proof. When $n_1 = 1$, we have

$$d_k = \dim Span_{\mathbb{R}} N_k = \left\{ \prod_{2\leq t\leq n} (x_1 x_t - y_1 y_t)^{h_t} | \sum_{2\leq t\leq n} h_t = k, h_t \in \mathbb{N} \right\} = \binom{n-1+k-1}{k} \approx c_0 k^{n-2},$$

for some positive constant c_0. The case for $n_1 = n-1$ is dual to the previous case.

When $n_1 = 2 < n-1$, we have

$$\begin{aligned} d_k &= \dim Span_{\mathbb{R}} N_k \\ &= \dim Span_{\mathbb{R}} \left\{ \prod_{\substack{1\leq i\leq 2\\ 3\leq t\leq n}} (x_i x_t - y_i y_t)^{h_{it}} | \sum_{\substack{1\leq i\leq 2\\ 3\leq t\leq n}} h_{it} = k, h_{it} \in \mathbb{N} \right\} \\ &\geq \dim Span_{\mathbb{R}} \left\{ \prod_{3\leq t\leq n} (x_1 x_t)^{h_{1t}} \prod_{3\leq t\leq n} (y_2 y_t)^{h_{2t}} | \sum_{\substack{1\leq i\leq 2\\ 3\leq t\leq n}} h_{it} = k, h_{it} \in \mathbb{N} \right\} \\ &\approx c_{11} k^{2n-5}. \end{aligned}$$

On the other hand, we have $d_k = \dim Span_{\mathbb{R}} N_k \leq c_{12} k^{n_1(n-n_1)-1} = c_{12} k^{2n-5}$, for some positive constant c_{12}. So we must have $d_k = \dim Span_{\mathbb{R}} N_k \approx c_1 k^{2n-5}$, for some positive constant c_1. The case for $1 < n_1 = n-2$ is dual to the previous case.

When $n_1 = 3 < n = 6$, we have

$$
\begin{aligned}
d_k &= \dim Span_{\mathbb{R}} N_k \\
&= \dim Span_{\mathbb{R}} \left\{ \prod_{\substack{1\le i\le 3\\ 4\le t\le 6}} (x_i x_t - y_i y_t)^{h_{it}} \mid \sum_{\substack{1\le i\le 3\\ 4\le t\le 6}} h_{it} = k, h_{it} \in \mathbb{N} \right\} \\
&\ge \dim Span_{\mathbb{R}} \Big\{ (x_1x_4)^{h_{14}} (x_1x_5)^{h_{15}} (x_2x_6)^{h_{26}} (x_2x_5)^{h_{25}} (x_3x_6)^{h_{36}} \\
&\qquad\qquad \cdot (y_1y_6)^{h_{16}} (y_2y_4)^{h_{24}} (y_3y_4)^{h_{34}} (y_3y_5)^{h_{35}} \mid \sum_{\substack{1\le i\le 3\\ 4\le t\le n}} h_{it} = k, h_{it} \in \mathbb{N} \Bigg\} \\
&= \dim Span_{\mathbb{R}} \Big\{ \Big((x_1)^{h_{14}+h_{15}} (x_2)^{h_{25}+h_{26}} (x_3)^{h_{36}} y_4^{h_{24}+h_{34}} y_5^{h_{35}} y_6^{h_{16}} \Big) \\
&\qquad \cdot \Big((x_4)^{h_{14}} (x_5)^{h_{15}+h_{25}} (x_6)^{h_{26}+h_{36}} (y_2)^{h_{24}} (y_3)^{h_{34}+h_{35}} y_1^{h_{16}} \Big) \mid \sum_{\substack{1\le i\le 3\\ 4\le t\le n}} h_{it} = k, h_{it} \in \mathbb{N} \Bigg\} \\
&\approx c_{21} k^8.
\end{aligned}
$$

On the other hand, we have $d_k = \dim Span_{\mathbb{R}} N_k \le c_{22} k^{n_1(n-n_1)-1} = c_{22}k^8$, for some positive constant c_{22}. So we must have $d_k = \dim Span_{\mathbb{R}} N_k \approx c_2 k^8 = c_2 k^{2n-4}$, for some positive constant c_2.

When $3 = n_1 < n - 3$, we have

$$\begin{aligned}
d_k &= \dim Span_{\mathbb{R}} N_k \\
&= \dim Span_{\mathbb{R}} \left\{ (\prod_{\substack{1\leq i\leq 3\\ 4\leq t\leq 6}} (x_i x_t - y_i y_t)^{h_{it}})(\prod_{\substack{1\leq i\leq 3\\ 7\leq t\leq n}} (x_i x_t - y_i y_t)^{h_{it}}) | \sum_{\substack{1\leq i\leq 3\\ 4\leq t\leq n}} h_{it} = k, h_{it} \in \mathbb{N} \right\} \\
&\geq \dim Span_{\mathbb{R}} \left\{ (\prod_{\substack{1\leq i\leq 3\\ 4\leq t\leq 6}} (x_i x_t - y_i y_t)^{h_{it}})(\prod_{7\leq t\leq n} (x_3 x_t)^{h_{3t}} (y_1 y_t)^{h_{1t}} (y_2 y_t)^{h_{2t}}) \right. \\
&\qquad \left. | \sum_{\substack{1\leq i\leq 3\\ 4\leq t\leq n}} h_{it} = k, h_{it} \in \mathbb{N} \right\} \\
&\geq \dim Span_{\mathbb{R}} \left\{ \left((x_1)^{h_{14}+h_{15}} (x_2)^{h_{25}+h_{26}} (x_3)^{h_{36}+\sum h_{3t}} y_4^{h_{24}+h_{34}} y_5^{h_{35}} y_6^{h_{16}} \prod_{7\leq t\leq n} y_t^{h_{1t}+h_{2t}} \right) \right. \\
&\qquad \cdot \left((x_4)^{h_{14}} (x_5)^{h_{15}+h_{25}} (x_6)^{h_{26}+h_{36}} (\prod_{7\leq t\leq n} x_t^{h_{3t}}) (y_2)^{h_{24}+\sum h_{2t}} (y_3)^{h_{34}+h_{35}} y_1^{h_{16}+\sum h_{it}} \right) \\
&\qquad \left. | \sum_{\substack{1\leq i\leq 3\\ 4\leq t\leq n}} h_{it} = k, h_{it} \in \mathbb{N} \right\} \\
&\approx c_{31} k^{2n-3}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d_k &= \dim Span_{\mathbb{R}} N_k \\
&\leq \dim Span_{\mathbb{R}} \left\{ \prod_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} (x_i x_t)^{p_{it}} \prod_{\substack{1\leq i\leq n_1\\ n_1+1\leq t\leq n}} (y_i y_t)^{q_{it}} | \sum p_{it} + \sum q_{it} = k \right\} \\
&\approx c_{32} k^{2n-3},
\end{aligned}$$

for some positive constant c_{32}. So we must have $d_k = \dim Span_{\mathbb{R}} N_k \approx c_3 k^{2n-3}$, for some positive constant c_3. The case for $3 < n_1 = n - 3$ is dual to the previous case.

When $3 < n_1 < n - 3$, we can use the same inductive argument with the above case and get

$$d_k = \dim Span_{\mathbb{R}} N_k \approx c_3 k^{2n-3},$$

for some positive constant c_3. □

Proposition 4.3. *Let* $k \in \mathbb{N}$. *Suppose* $2 < n_1 + 1 \leq n_2 < n - 1$ *and we denote*

$$N'_k = \Bigg\{ \prod (x_i x_s)^{p_{is}} \prod (y_s y_t)^{q_{st}} \prod (x_i x_t - y_i y_t)^{h_{it}} | \sum_{\substack{1 \leq i \leq n_1 \\ n_1+1 \leq s \leq n_2}} p_{is} + \sum_{\substack{n_1+1 \leq s \leq n_2 \\ n_2+1 \leq t \leq n}} q_{st} + \sum_{\substack{1 \leq i \leq n_1 \\ n_2+1 \leq t \leq n}} h_{it} = k \Bigg\},$$

then

$$d'_k = \dim Span_{\mathbb{R}} N'_k \approx ck^{2n-3},$$

for some constant c.

Proof. When $n_1 = 2 < n_2 < n - 1$, then from Prop 4.2 we have

$$\begin{aligned}
d'_k &= \dim Span_{\mathbb{R}} \left\{ \prod (x_i x_s)^{p_{is}} \prod (y_s y_t)^{q_{st}} \prod (x_i x_t - y_i y_t)^{h_{it}} | \sum p_{is} + \sum q_{st} + \sum h_{it} = k \right\} \\
&\geq \dim Span_{\mathbb{R}} \Bigg\{ \Bigg((x_2)^{h_{2,n_2+1} + \sum p_{2s}} (x_1)^{\sum p_{1s}} (\prod_{n_2+2 \leq t \leq n} (x_1)^{h_{1t}})(\prod_{n_2+2 \leq t \leq n} (y_t)^{h_{2t}})(y_t)^{\sum q_{st}} (y_{n_2+1})^{h_{1,n_2+1}} \Bigg) \\
&\qquad \cdot \Bigg(\prod (x_s)^{p_{1s}+p_{2s}} (x_{n_2+1})^{h_{2,n_2+1}} (\prod_{n_2+2 \leq t \leq n} (x_t)^{h_{1t}})(\prod (y_s)^{q_{st}})(y_1)^{h_{1,n_2+1}} \prod_{n_2+2 \leq t \leq n} (y_2)^{h_{2t}} \Bigg) \\
&\qquad | \sum p_{is} + \sum q_{st} + \sum h_{it} = k \Bigg\} \\
&\approx c_0 k^{2n-3}, \textit{ for some constant } c_0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
d'_k =& \dim Span_{\mathbb{R}} N'_k \\
\leq& \dim Span_{\mathbb{R}} \left\{ \prod (x_i x_s)^{p_{is}} \prod (y_s y_t)^{q_{st}} \prod (x_i x_t)^{l_{it}} \prod (y_i y_t)^{f_{it}} | \right. \\
& \left. \sum h_{it} + \sum p_{it} + \sum l_{it} + \sum f_{it} = k \right\} \\
=& \dim Span_{\mathbb{R}} \left\{ \prod_{\substack{1 \leq i \leq n_1 \\ n_1+1 \leq s_0 \leq n}} (x_i x_{s_0})^{p_{is_0}} \prod_{\substack{1 \leq i_0 \leq n_2 \\ n_2+1 \leq t \leq n}} (y_{i_0} y_t)^{q_{i_0 t}} | \sum p_{is_0} + \sum q_{i_0 t} = k \right\} \qquad (4.7) \\
\approx& c_{00} k^{2n-3}, \textit{ for some constant } c_{00}.
\end{aligned}$$

So we must have $d'_k = \dim Span_{\mathbb{R}} N'_k \approx ck^{2n-3}$, for some positive constant c.

From Prop 4.2, we can use the similar argument to compute the other cases. And for these cases we still have $d'_k = \dim Span_{\mathbb{R}} N'_k \approx ck^{2n-3}$, for some positive constant c. □

Next, we will compute the Gelfand-Kirillov dimensions of our modules in a case-by-case way.

4.1. **Case 1.** $n_1+1<n_2<n$**.** In this case we have:

$$E_{r,i}|_{\mathscr{B}}=-x_i\partial_{x_r}-y_i\partial_{y_r} \quad \text{for } 1\le i<r\le n_1, \tag{4.8}$$

$$E_{s,i}|_{\mathscr{B}}=-x_ix_s-y_i\partial_{y_s} \quad \text{for } i\in\overline{1,n_1},\ s\in\overline{n_1+1,n_2}, \tag{4.9}$$

$$E_{t,i}|_{\mathscr{B}}=-x_ix_t+y_iy_t \quad \text{for } i\in\overline{1,n_1},\ t\in\overline{n_2+1,n}, \tag{4.10}$$

$$E_{s,j}|_{\mathscr{B}}=x_s\partial_{x_j}-y_j\partial_{y_s} \quad \text{for } n_1<j<s\le n_2, \tag{4.11}$$

$$E_{t,s}|_{\mathscr{B}}=x_t\partial_{x_s}+y_sy_t \quad \text{for } s\in\overline{n_1+1,n_2},\ t\in\overline{n_2+1,n}, \tag{4.12}$$

$$E_{t,p}|_{\mathscr{B}}=x_t\partial_{x_p}+y_t\partial_{y_p} \quad \text{for } n_2+1\le p<t\le n. \tag{4.13}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$.

From Luo-Xu [17] we know that for $m_1,m_2\in\mathbb{N}$ with $m_1+m_2\ge n_2-n_1-1$, $\mathscr{H}_{\langle -m_1,-m_2\rangle}$ has a highest-weight vector $v_\lambda=x_{n_1}^{m_1}y_{n_2+1}^{m_2}$ of weight $\lambda=m_1\lambda_{n_1-1}-(m_1+1)\lambda_{n_1}-(m_2+1)\lambda_{n_2}+m_2(1-\delta_{n_2,n-1})\lambda_{n_2+1}$. Then

$$E_{s,j}|_{\mathscr{B}}v_\lambda=0 \quad \text{for } n_1<j<s\le n_2, \tag{4.14}$$

$$E_{n_1,i}|_{\mathscr{B}}v_\lambda=-x_i\partial_{x_{n_1}}(x_{n_1}^{m_1}y_{n_2+1}^{m_2}), \quad \text{for } 1\le i<n_1, \tag{4.15}$$

$$E_{r,i}|_{\mathscr{B}}v_\lambda=0 \quad \text{for } 1\le i<r<n_1, \tag{4.16}$$

$$E_{t,p}|_{\mathscr{B}}v_\lambda=0 \quad \text{for } n_2+1<p<t\le n, \tag{4.17}$$

$$E_{t,n_2+1}|_{\mathscr{B}}v_\lambda=y_t\partial_{y_{n_2+1}}(x_{n_1}^{m_1}y_{n_2+1}^{m_2}) \quad \text{for } n_2+1<t\le n. \tag{4.18}$$

Let $\mathfrak{g}_1$ be the subalgebra of $\mathfrak{sl}(n,\mathbb{F})$ spanned by the following set:

$$\{E_{s,j},E_{r,i},E_{t,p}|n_1<j<s\le n_2, 1\le i<r\le n_1, n_2+1\le p<t\le n\}.$$

Let $\mathfrak{g}_2$ be the subalgebra of $\mathfrak{sl}(n,\mathbb{F})$ spanned by the following set:

$$\{E_{s,i},E_{t,i},E_{t,s}|i\in\overline{1,n_1}, s\in\overline{n_1+1,n_2}, t\in\overline{n_2+1,n}\}.$$

So we get $\mathcal{U}(\mathfrak{g}_-)=\mathcal{U}(\mathfrak{g}_2)\mathcal{U}(\mathfrak{g}_1)$.

Observe that

$$\begin{aligned}
&\mathcal{U}(\mathfrak{g}_1)v_\lambda\\
=&Span_{\mathbb{R}}\{\prod_{i=1}^{n_1-1}E_{n_1,i}^{k_i}\prod_{t=n_2+2}^{n}E_{t,n_2+1}^{l_t}v_\lambda|k_i,l_t\in\mathbb{N}\}\\
=&Span_{\mathbb{R}}\{\prod_{i=1}^{n_1-1}x_i^{k_i}x_{n_1}^{(m_1-\sum_{i=1}^{n_1-1}k_i)}\prod_{t=n_2+2}^{n}y_t^{l_t}y_{n_2+1}^{(m_2-\sum_{t=n_2+2}^{n}l_t)}|k_i,l_t\in\mathbb{N}\}\\
=&Span_{\mathbb{R}}\{\prod_{i=1}^{n_1}x_i^{k_i}\prod_{t=n_2+1}^{n}y_t^{l_t}|\sum_{i=1}^{n_1}k_i=m_1,\sum_{t=n_2+1}^{n}l_t=m_2\}
\end{aligned}$$

We denote this space by M_0. Then M_0 is a subspace of the space spanned by the homogeneous polynomials of degree m_1+m_2 in $\mathbb{F}[x_1,...,x_{n_1},y_{n_2+1},...,y_n]$. So M_0 is finite-dimensional.

Thus

$$\mathcal{U}(\mathfrak{g}_-)v_\lambda=\mathcal{U}(\mathfrak{g}_2)M_0.$$

Now we take any base element $u_0 = \prod_{i=1}^{n_1} x_i^{k_i} \prod_{t=n_2+1}^{n} y_t^{l_t} \in M_0$. Let k be any positive integer. We want to compute $\dim(\mathcal{U}_k(\mathfrak{g}_2)M_0)$, and get the Gelfand-Kirillov dimension of $\mathcal{U}(\mathfrak{g}_2)M_0$.

We denote

$$N_0(k) = \left\{ \left(\prod E_{s,i}^{p_{si}} \prod E_{t,i}^{h_{ti}} \prod E_{t,s}^{q_{ts}}\right) u_0 | p_{si}, h_{ti}, q_{ts} \in \mathbb{N}, \right.$$
$$\left. \sum_{\substack{1\le i\le n_1\\ n_1+1\le s\le n_2}} p_{si} + \sum_{\substack{1\le i\le n_1\\ n_2+1\le t\le n}} h_{ti} + \sum_{\substack{n_1+1\le s\le n_2\\ n_2+1\le t\le n}} q_{ts} = k \right\}.$$

From the definition we know

$$\begin{aligned}
&\left(\prod E_{s,i}^{p_{si}} \prod E_{t,i}^{h_{ti}} \prod E_{t,s}^{q_{ts}}\right) u_0\\
&= \left(\prod(-x_i x_s - y_i\partial_{y_s})^{p_{si}} \prod(-x_i x_t + y_i y_t)^{h_{ti}} \prod(y_s y_t)^{q_{ts}}\right) \prod_{i=1}^{n_1} x_i^{k_i} \prod_{t=n_2+1}^{n} y_t^{l_t}\\
&= \left(\prod(-x_i x_s)^{p_{si}} \prod(-x_i x_t + y_i y_t)^{h_{ti}} \prod(y_s y_t)^{q_{ts}}\right) \prod_{i=1}^{n_1} x_i^{k_i} \prod_{t=n_2+1}^{n} y_t^{l_t}\\
&\quad + \text{lower degree part of } y_s.
\end{aligned}$$

Then we must have

$$\dim Span_{\mathbb{R}} N_0(m) \ge d'_m.$$

Using the same idea with inequality 4.7, we can also get

$$\dim Span_{\mathbb{R}} N_0(m) \le d'_m.$$

Thus $\dim Span_{\mathbb{R}} N_0(m) = d'_m$.
Then using proposition 4.3, we can get

$$\begin{aligned}
&\dim Span_{\mathbb{R}}\Big(\bigcup_{0\le m\le k} N_0(m)\Big)\\
&= \sum_{0\le m\le k} d'_m\\
&= \begin{cases} b_1 \sum\limits_{0\le m\le k} m^{2n-4}, & \text{if } 2 = n_1+1 < n_2 < n \text{ or } n_1+1 < n_2 = n-1\\ c_1 \sum\limits_{0\le m\le k} m^{2n-3}, & \text{if } 2 < n_1+1 < n_2 < n-1 \end{cases}\\
&= \begin{cases} bk^{2n-3}, & \text{if } 2 = n_1+1 < n_2 < n \text{ or } n_1+1 < n_2 = n-1\\ ck^{2n-2}, & \text{if } 2 < n_1+1 < n_2 < n-1. \end{cases}
\end{aligned}$$

We know

$$\dim Span_{\mathbb{R}}\Big(\bigcup_{0\le m\le k} N_0(m)\Big) \le \dim(\mathcal{U}_k(\mathfrak{g}_2)M_0) \le \dim M_0 \dim Span_{\mathbb{R}}\Big(\bigcup_{0\le m\le k} N_0(m)\Big).$$

Then from the definition, we know that the Gelfand-Kirillov dimension of $\mathcal{U}(\mathfrak{g}_-)v_\lambda$ is

$$d = \begin{cases} 2n-3, & \text{if } 2 = n_1+1 < n_2 < n \text{ or } n_1+1 < n_2 = n-1\\ 2n-2, & \text{if } 2 < n_1+1 < n_2 < n-1. \end{cases}$$

From Luo-Xu [17] we know that for $m_1, m_2 \in \mathbb{N}$ with $m_2 - m_1 \geq n_2 - n_1 - 1$, $\mathscr{H}_{\langle m_1, -m_2\rangle}$ has a highest-weight vector $x_{n_1+1}^{m_1} y_{n_2+1}^{m_2}$ of weight $-(m_1+1)\lambda_{n_1} + m_1\lambda_{n_1+1} - (m_2+1)\lambda_{n_2} + m_2(1-\delta_{n_2,n-1})\lambda_{n_2+1}$. For $m_1, m_2 \in \mathbb{N}$ with $m_1 - m_2 \geq n_2 - n_1 - 1$, $\mathscr{H}_{\langle -m_1, m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_2}^{m_2}$ of weight $m_1\lambda_{n_1-1} - (m_1+1)\lambda_{n_1} + m_2\lambda_{n_2-1} - (m_2+1)\lambda_{n_2}$. The arguments for these two cases are similar to the above case, and we have the same Gelfand-Kirillov dimension

4.2. **Case 2.** $n_1 + 1 < n_2 = n$. In this case we have:

$$E_{r,i}|_{\mathscr{B}} = -x_i\partial_{x_r} - y_i\partial_{y_r} \qquad \text{for } 1 \leq i < r \leq n_1, \tag{4.19}$$

$$E_{s,i}|_{\mathscr{B}} = -x_i x_s - y_i\partial_{y_s} \qquad \text{for } i \in \overline{1, n_1},\ s \in \overline{n_1+1, n}, \tag{4.20}$$

$$E_{s,j}|_{\mathscr{B}} = x_s\partial_{x_j} - y_j\partial_{y_s} \qquad \text{for } n_1 < j < s \leq n. \tag{4.21}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$.

From Luo-Xu [17] we know that for $m_1, m_2 \in \mathbb{N}$, $\mathscr{H}_{\langle m_1, m_2\rangle}$ has a highest-weight vector $x_{n_1+1}^{m_1} y_n^{m_2}$ of weight $-(m_1+1)\lambda_{n_1} + m_1\lambda_{n_1+1} + m_2\lambda_{n-1}$. For $m_1, m_2 \in \mathbb{N}$ with $m_1 \leq n - n_1 - 2$ or $m_2 - m_1 \leq n_1 - n + 1$, $\mathscr{H}_{\langle -m_1, m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_n^{m_2}$ of weight $m_1\lambda_{n_1-1} - (m_1+1)\lambda_{n_1} + m_2\lambda_{n-1}$.

The arguments for these two cases are similar to case 1, and from proposition 4.1 we have the Gelfand-Kirillov dimension equal to

$$n - 1.$$

4.3. **Case 3.** $n_1 + 1 = n_2 < n$. In this case we have:

$$E_{r,i}|_{\mathscr{B}} = -x_i\partial_{x_r} - y_i\partial_{y_r} \qquad \text{for } 1 \leq i < r \leq n_1, \tag{4.22}$$

$$E_{n_1+1,i}|_{\mathscr{B}} = -x_i x_{n_1+1} - y_i\partial_{y_{n_1+1}} \qquad \text{for } i \in \overline{1, n_1}, \tag{4.23}$$

$$E_{t,i}|_{\mathscr{B}} = -x_i x_t + y_i y_t \qquad \text{for } i \in \overline{1, n_1},\ t \in \overline{n_1+2, n}, \tag{4.24}$$

$$E_{t,n_1+1}|_{\mathscr{B}} = x_t\partial_{x_{n_1+1}} + y_{n_1+1} y_t \qquad \text{for } t \in \overline{n_1+2, n}, \tag{4.25}$$

$$E_{t,p}|_{\mathscr{B}} = x_t\partial_{x_p} + y_t\partial_{y_p} \qquad \text{for } n_1 + 2 \leq p < t \leq n. \tag{4.26}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$.

From Luo-Xu [17] we know that for $m_1, m_2 \in \mathbb{N}$, $\mathscr{H}_{\langle -m_1, -m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_1+2}^{m_2}$ of weight $m_1\lambda_{n_1-1} - (m_1+1)\lambda_{n_1} - (m_2+1)\lambda_{n_1+1} + m_2(1-\delta_{n_1,n-2})\lambda_{n_1+2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_2 - m_1 \geq 0$, $\mathscr{H}_{\langle m_1, -m_2\rangle}$ has a highest-weight vector $x_{n_1+1}^{m_1} y_{n_1+2}^{m_2}$ of weight $-(m_1+1)\lambda_{n_1} + (m_1 - m_2 - 1)\lambda_{n_1+1} + m_2(1-\delta_{n_1,n-2})\lambda_{n_1+2}$. For $m_1, m_2 \in \mathbb{N}$ with $m_1 - m_2 \geq 0$, $\mathscr{H}_{\langle -m_1, m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_1+1}^{m_2}$ of weight $m_1\lambda_{n_1-1} + (m_2 - m_1 - 1)\lambda_{n_1} - (m_2+1)\lambda_{n_1+1}$.

The arguments for these three cases are similar to case 1, and we have the Gelfand-Kirillov dimension equal to

$$d = \begin{cases} 2n-3, & \text{if } 2 = n_1 + 1 = n_2 < n \text{ or } n_1 + 1 = n_2 = n - 1 \\ 2n-2, & \text{if } 2 < n_1 + 1 = n_2 < n - 1. \end{cases}$$

4.4. **Case 4.** $n_1 + 1 = n_2 = n$. In this case we have:

$$E_{r,i}|_{\mathscr{B}} = -x_i\partial_{x_r} - y_i\partial_{y_r} \qquad \text{for } 1 \leq i < r \leq n-1, \tag{4.27}$$

$$E_{n,i}|_{\mathscr{B}} = -x_i x_n - y_i\partial_{y_n} \qquad \text{for } i \in \overline{1, n-1}. \tag{4.28}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$. In this case, we denote

$$\eta = \sum_{i=1}^{n-1} y_i\partial_{x_i} + x_n y_n. \tag{4.29}$$

From Luo-Xu [17] we know that for $m_1, m_2 \in \mathbb{N}$ with $m_2 \le m_1$, $\mathscr{H}_{\langle -m_1, m_2\rangle}$ has a highest-weight vector $x_{n-1}^{m_1} y_n^{m_2}$ of weight $m_1\lambda_{n-2} + (m_2 - m_1 - 1)\lambda_{n-1}$. Moreover, $\mathscr{H}_{\langle m,0\rangle}$ has a highest-weight vector x_{n-1}^{m} of weight $m\lambda_{n-2} - (m+1)\lambda_{n-1}$ for $m \in \mathbb{Z}$.

The arguments for these two cases are similar to case 2, and we have the Gelfand-Kirillov dimension equal to

$$n-1.$$

For $m_1, m_2 \in \mathbb{N}+1$, $\mathscr{H}_{\langle m_1, m_2\rangle}$ has a highest-weight vector $\eta^{m_1+m_2}(x_{n-1}^{m_2} y_n^{-m_1})$ of weight $m_2\lambda_{n-2} + (m_1 - m_2 - 1)\lambda_{n-1}$. Then similar to the the arguments in case 2, we have the Gelfand-Kirillov dimension equal to

$$n-1.$$

4.5. **Case 5.** $n_1 = n_2 < n-1$**.** In this case we have:

$$E_{r,i}|_{\mathscr{B}} = -x_i\partial_{x_r} - y_i\partial_{y_r} \qquad \text{for } 1 \le i < r \le n_1, \tag{4.30}$$

$$E_{t,i}|_{\mathscr{B}} = -x_i x_t + y_i y_t \qquad \text{for } i \in \overline{1, n_1},\ t \in \overline{n_1+1, n}, \tag{4.31}$$

$$E_{t,p}|_{\mathscr{B}} = x_t\partial_{x_p} + y_t\partial_{y_p} \qquad \text{for } n_1+1 \le p < t \le n. \tag{4.32}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$.

In this case, recall

$$\zeta_1 = x_{n_1-1}y_{n_1} - x_{n_1}y_{n_1-1}, \quad \zeta_2 = x_{n_1+1}y_{n_1+2} - x_{n_1+2}y_{n_1+1}. \tag{4.33}$$

From Luo-Xu [17] we know that for $m_1, m_2 \in \mathbb{N}$, the subspace $\mathscr{H}_{\langle -m_1, -m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_1+1}^{m_2}$ of weight $m_1(1-\delta_{1,n_1})\lambda_{n_1-1} - (m_1+m_2+2)\lambda_{n_1} + m_2\lambda_{n_1+1}$. The subspace $\mathscr{H}_{\langle m_1+1, -m_2-m_1-1\rangle}$ has a highest-weight vector $y_{n_1+1}^{m_2}\zeta_2^{m_1+1}$ of weight $-(m_1+m_2+3)\lambda_{n_1} + m_2\lambda_{n_1+1} - (m_1+1)(1-\delta_{n_1,n-2})\lambda_{n_1+2}$. If $n_1 \ge 2$, the subspace $\mathscr{H}_{\langle -m_1-m_2-1, m_2+1\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}\zeta_1^{m_2+1}$ of weight $(m_2+1)\lambda_{n_1-2} - m_1\lambda_{n_1-1} - (m_1+m_2+3)\lambda_{n_1}$.

Then from Prop 4.2 we have the Gelfand-Kirillov dimension equal to

$$d = \begin{cases} n-1, & \text{if } 1 = n_1 = n_2 < n-1 \\ 2n-4, & \text{if } n_1 = n_2 = 2 < n-1 \text{ or } 1 < n_1 = n_2 = n-2 \\ 2n-3, & \text{if } n_1 = n_2 = 3 < n = 6 \\ 2n-2, & \text{if } 3 \le n_1 = n_2 \le n-3, n \ge 7. \end{cases}$$

4.6. **Case 6.** $n_1 = n_2 = n-1$**.** In this case we have:

$$E_{r,i}|_{\mathscr{B}} = -x_i\partial_{x_r} - y_i\partial_{y_r} \qquad \text{for } 1 \le i < r \le n-1, \tag{4.34}$$

$$E_{n,i}|_{\mathscr{B}} = -x_i x_n + y_i y_n \qquad \text{for } i \in \overline{1, n-1}. \tag{4.35}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$.

In this case, we denote

$$\zeta_1 = x_{n-2}y_{n-1} - x_{n-1}y_{n-2}. \tag{4.36}$$

From Luo-Xu [17] we know that for $m_1, m_2 \in \mathbb{N}$, the subspace $\mathscr{H}_{\langle -m_1, -m_2\rangle}$ has a highest-weight vector $x_{n_1}^{m_1} y_{n_1+1}^{m_2}$ of weight $m_1(1-\delta_{n,2})\lambda_{n-2} - (m_1+m_2+2)\lambda_{n-1}$. If $n \ge 3$, the subspace $\mathscr{H}_{\langle -m_1-m_2-1, m_2+1\rangle}$ has a highest-weight vector $x_{n_1}^{m_1}\zeta_1^{m_2+1}$ of weight $(m_2+1)(1-\delta_{n,3})\lambda_{n-3} - m_1\lambda_{n-2} - (m_1+m_2+3)\lambda_{n-1}$.

Then similar to the the arguments in case 2, we have the Gelfand-Kirillov dimension equal to

$$n-1.$$

4.7. **Case 7.** $n_1 = n_2 = n$**.** In this case we have:

$$E_{r,i}|_{\mathscr{B}} = -x_i\partial_{x_r} - y_i\partial_{y_r} \qquad \text{for } 1 \le i < r \le n \tag{4.37}$$

Then the above $E_{j,i}$ forms a basis for the subalgebra $\mathfrak{sl}(n,\mathbb{F})_-$.

From Luo-Xu [17] we know that the subspace $\mathscr{H}_{\langle -m_1-m_2,m_2\rangle}$ is finite dimensional and has a highest-weight vector $x_n^{m_1}\zeta_1^{m_2}$ of weight $m_2(1-\delta_{n,2})\lambda_{n-2}+m_1\lambda_{n-1}$. Obviously its Gelfand-Kirillov dimension is equal to 0.

This completes the proof of our formula.

5. Minimal Gelfand-Kirillov dimension module

Let M be an irreducible highest-weight module of $\mathfrak{sl}(n,\mathbb{F})$. We denote its Gelfand-Kirillov dimension by d_M. From Vogan [24] and Wang [26], we know that the minimal Gelfand-Kirillov dimension is $d_M = n-1$. The corresponding modules are called the minimal GK-dimension module.

From Luo-Xu [17] and our formula 1.8, we find the following result.

Corollary 5.1. *Let $n(\geq 2)$ be a positive integer. Fix $n_1, n_2 \in \overline{1,n}$ with $n_1 \le n_2$. The irreducible highest-weight $\mathfrak{sl}(n,\mathbb{F})$-module $\mathscr{H}_{\langle \ell_1,\ell_2\rangle}$ has the minimal Gelfand-Kirillov dimension if and only if*

(1) $n_1 < n_2 = n$.
(2) $n_1 = n_2 = 1$.
(3) $n_1 = n_2 = n-1$.

We denote $\mathscr{A} = \mathbb{F}[x_1,...,x_n]$. Fix $1 \le n_1 < n$. Changing operators $\partial_{x_r} \mapsto -x_r$ and $x_r \mapsto \partial_{x_r}$ in the canonical oscillator representation $E_{i,j}|\mathscr{A} = x_i\partial_j$ for $r \in \overline{1,n_1}$, we obtain the following non-canonical oscillator representation of $\mathfrak{sl}(n,\mathbb{F})$ determined by:

$$E_{i,j}|_{\mathscr{A}} = \begin{cases} -x_j\partial_{x_i} - \delta_{i,j} & \text{if } i,j \in \overline{1,n_1};\\ \partial_{x_i}\partial_{x_j} & \text{if } i \in \overline{1,n_1},\ j \in \overline{n_1+1,n};\\ -x_ix_j & \text{if } i \in \overline{n_1+1,n},\ j \in \overline{1,n_1};\\ x_i\partial_{x_j} & \text{if } i,j \in \overline{n_1+1,n}. \end{cases} \tag{5.1}$$

For any $k \in \mathbb{Z}$, we denote

$$\mathscr{A}_{\langle k\rangle} = \text{Span}\,\{x^\alpha \mid \alpha \in \mathbb{N}^{\,n};\ \sum_{r=n_1+1}^{n}\alpha_r - \sum_{i=1}^{n_1}\alpha_i = k\}.$$

It was presented by Howe [11] that for $m_1, m_2 \in \mathbb{N}$ with $m_1 > 0$, $\mathscr{A}_{\langle -m_1\rangle}$ is an irreducible highest-weight $\mathfrak{sl}(n,\mathbb{F})$-submodule with highest weight $m_1\lambda_{n_1-1} - (m_1+1)\lambda_{n_1}$ and $\mathscr{A}_{\langle m_2\rangle}$ is an irreducible highest-weight $\mathfrak{sl}(n,\mathbb{F})$-submodule with highest weight $-(m_2+1)\lambda_{n_1} + m_2(1-\delta_{n_1,n-1})\lambda_{n_1+1}$.

By a similar argument with our formula, we have the following corollary.

Corollary 5.2. *Fix $1 \le n_1 < n$. For $m_1, m_2 \in \mathbb{N}$ with $m_1 > 0$, the irreducible highest-weight module $\mathscr{A}_{\langle -m_1\rangle}$ and $\mathscr{A}_{\langle m_2\rangle}$ have the minimal Gelfand-Kirillov dimension $(n-1)$.*

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REFERENCES

[1] Benkart, G., Britten, D. J., Lemire, F. W.: Modules with bounded weight multiplicities for simple Lie algebras, *Math. Z.*, **225**, 333-353 (1997) 2, 3, 4

[2] Borho, W., Kraft, H.: Über die Gelfand-Kirillov dimension. *Math. Ann.*, **220**, 1-24 (1976) 4

[3] Britten, D. J., Hooper, J., Lemire, F. W.: Simple C_n modules with multiplicities 1 and applications, *Canad. J. Phys.*, **72**, 326-335 (1994) 1

[4] Britten, D. J., Khomenko, O., Lemire, F. W., Mazorchuk, V.: Complete reducibility of torsion free C_n-modules of finite degree, *J. Algebra*, **276**, 129-142 (2004) 1

[5] Britten, D. J., Lemire, F. W.: On Modules of bounded weight multiplicities for symplectic algebras, *Trans. Amer. Math. Soc.*, **351**, 3413-3431 (1999) 2

[6] Futony, V.: The weight representations of semisimple finite-dimensional Lie algebras, *Ph.D. Thesis*, Kiev University, 1987. 1

[7] Fernando, S. L.: Lie algebra modules with finite-dimensional weight spaces. I, *Trans. Amer. Math. Soc.*, **322**, 757-781 (1990) 1

[8] Gelfand, I. G., Kirillov, A. A.: Sur les corps liés aux algébres enveloppantes des algébres de Lie, *Publ. Math. IHES.*, **31**, 5-19 (1966) 1

[9] Grantcharov, D., Serganova, V.: Category of $\mathfrak{sp}(2n)$-modules with bounded weight multiplicities, *Mosc. Math. J.*, **6**, 119-134 (2006) 1

[10] Grantcharov, D., Serganova, V.: Cuspidal representations of $\mathfrak{sl}(n+1)$, *Adv. Math.*, **224**, 1517-1547 (2010) 1

[11] Howe, R.: Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in: The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proc., Vol. 8, pp. 1-182, Bar-Ilan Univ., Ramat Gan, 1995 16

[12] Jantzen, J. C.: Einhüllende Algebren halbeinfacher Lie-Algebren. (German) [Enveloping algebras of semisimple Lie algebras], Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 3, Springer-Verlag, Berlin, 1983 4

[13] Joseph, A.: Gelfand-Kirillov dimension for the annihilators of simple quotients of Verma modules, *J. London Math. Soc.* (2), **18**, 50-60 (1978) 1

[14] Knapp, A. W.: Lie groups beyond an Introduction, 2nd ed., Progress in Mathematics, Vol. 140, Birkhäuser Boston, Boston, 2002 5

[15] Krause, G., Lenagan, T. H.: Growth of Algebras and Gelfand-Kirillov dimension, Research Notes in Math. Pitman Adv. Publ. Program, Vol. 116, 1985 4

[16] Lepowsky, J.: Generalized Verma Modules, the Cartan-Helgason Theorem, and the Harish-Chandra Homomorphism, *J. Algebra*, **49**, 470-495 (1977)

[17] Luo, C., Xu, X.: $\mathbb{Z}^2$-graded oscillator representations of $\mathfrak{sl}(n)$, *Commun. Algebra,*, **41**, 3147-3173 (2013) 2, 3, 4, 6, 12, 14, 15, 16

[18] Mathieu, O.: Classification of irreducible weight modules, *Ann. Inst. Fourier (Grenoble)*, **50**, 537-592 (2000) 1

[19] Melnikov, A.: Irreducibility of the associated varieties of simple highest weight modules in $\mathfrak{sl}(n)$, *C. R. Acad. Sci. Paris Sér. I Math.*, **316**, 53-57 (1993)

[20] Nishiyama, K., Ochiai, H., Taniguchi, K., Yamashita, H., Kato, S.: Nilpotent orbits, associated cycles and Whittaker models for highest weight representations, *Astérisque*, **273**, 1-163 (2001) 4

[21] Riordan, J.: Combinatorial identities, John Wiley & Sons, Inc., New York-London-Sydney, 1968

[22] Sun, B.: Lowest weight modules of $\widetilde{Sp_{2n}}(\mathbb{R})$ of minimal Gelfand-Kirillov dimension, *J. Algebra*, **319**, 3062-3074 (2008) 2

[23] Vogan, D.: Gelfand-Kirillov dimension for Harish-Chandra modules, *Invent.Math.*, **48**, 75-98 (1978) 1, 4

[24] Vogan, D.: Singular unitary representations, in: Noncommutative harmonic analysis and Lie groups, Lecture Notes in Mathamatics, Vol. 880, pp. 506-535, Springer, Berlin-New York, 1981 1, 16

[25] Vogan, D.: Associated varieties and unipotent representations, in: Harmonic Analysis on Reductive Groups, Progr. Math. Vol. 101, pp. 315–388, Birkhäuser, Boston, 1991 4

[26] Wang, W.: Dimension of a minimal nilpotent orbit, *Proc. Amer. Math. Soc.*, **127**, 935-936 (1999) 1, 16

[27] Xu, X.: Flag partial differential equations and representations of Lie algebras, *Acta. Appl. Math.*, **102**, 249–280 (2008) 2

[28] Yamashita, H.: Criteria for the finiteness of restriction of $U(\mathfrak{g})$-modules to subalgebras and applications to Harish-Chandra modules: a study in relation to the associated varieties, *J. Funct. Anal.*, **121**, 296-329 (1994) 5

[29] Zariski, O., Samuel, P.: Commutative algebra, Vol. II, Springer-Verlag, New York/Berlin(World Publishing Corporation, China), 1975 5

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